National
Qualifications
2016

2016 Mathematics

Advanced Higher

Finalised Marking Instructions

Question			Generic Scheme	Illustrative Scheme	Max Mark
1.	(a)		•[1] evidence of use of product rule [1,2]	•[1] $(\ldots)\tan^{-1}2x + x(\ldots)$	3
			•[2] one resultant term of the product correct	•[2] $1.\tan^{-1}2x$ or $x.\dfrac{1}{1+(2x)^2}.2$	
			•[3] complete differentiation [3]	•[3] $\tan^{-1}2x + \dfrac{2x}{1+4x^2}$	

Notes:

1. Evidence for the award of •[1] should take the form $f(x)\times(\ldots)+g(x)\times(\ldots)$.
2. For a candidate who interprets $\tan^{-1}2x$ as $(\tan 2x)^{-1}$ •[3] is not available.
3. Accept $(2x)^2$ when awarding •[3].

Commonly Observed Responses:

	(b)		•[4] evidence of use of quotient or product rule and one term of numerator correct	•[4] $(-2x)(1+4x^2)-$	3
			•[5] complete differentiation correctly	•[5] $\dfrac{\ldots(1-x^2).8x}{(1+4x^2)^2}$	
			•[6] simplify answer [4,5]	•[6] $-\dfrac{10x}{(1+4x^2)^2}$ or $\dfrac{-10x}{(1+4x^2)^2}$	

Notes:

4. Where a candidate uses the product rule, simplification to $-\dfrac{10x}{(1+4x^2)^2}$ **or** $-10x(1+4x^2)^{-2}$ will be required in order to obtain •[6].
5. Incorrect working subsequent to a correct answer should be penalised in this instance eg an incorrect expansion of the denominator.

Commonly Observed Responses:

	(c)		•[7] correct derivatives	•[7] 6 and $\sin t$	2
			•[8] find $\dfrac{dy}{dx}$	•[8] $\dfrac{1}{6}\sin t$	

Notes

Commonly Observed Responses:

<table>
<tr><th colspan="3">Question</th><th>Generic Scheme</th><th>Illustrative Scheme</th><th>Max Mark</th></tr>
<tr><td>2.</td><td>(a)</td><td></td><td>•1 interpret geometric series

•2 evidence of strategy 1,2

•3 value 2</td><td>•1 $ar=108$ and $ar^4=4$

•2 $\dfrac{ar^4}{ar}$
$r^3=\dfrac{1}{27}$

•3 $r=\dfrac{1}{3}$</td><td>3</td></tr>
<tr><td colspan="6">Notes:
1. For •2 accept $r^3=\dfrac{1}{27}$.
2. For a statement of the answer only, award •1 and •3. To earn •2 there must be evidence of a strategy eg $108\rightarrow36\rightarrow12\rightarrow4$ gives $r=\dfrac{1}{3}$.</td></tr>
<tr><td colspan="6">Commonly Observed Responses:</td></tr>
<tr><td></td><td>(b)</td><td></td><td>•4 know condition 3,4</td><td>•4 $-1<\dfrac{1}{3}<1$</td><td>1</td></tr>
<tr><td colspan="6">Notes:
3. For •4 $\dfrac{1}{3}$ may be replaced with a letter consistent with their answer to (a). However, in the case where a candidate obtains a value in (a) outside the open interval $(-1,1)$ •4 will only be available where they also acknowledge that there is no sum to infinity.
4. Only award •4 for a strict inequality, whether it is expressed algebraically or in words.</td></tr>
<tr><td colspan="6">Commonly Observed Responses:</td></tr>
<tr><td></td><td>(c)</td><td></td><td>•5 calculate the first term

•6 value 5,6</td><td>•5 $a=324$

•6 $\dfrac{324}{1-\frac{1}{3}}$ or equivalent leading to 486</td><td>2</td></tr>
<tr><td colspan="6">Notes:
5. For an incorrect value in (a) •6 will only be available provided the value satisfies the condition for convergence.
6. Where a candidate has used $S_\infty=\dfrac{a(1-r^\infty)}{1-r}$ full credit is available.</td></tr>
<tr><td colspan="6">Commonly Observed Responses:</td></tr>
</table>

Question			Generic Scheme	Illustrative Scheme	Max Mark
3.			•[1] state general term [2]	•[1] ${}^{13}C_r \left(\frac{3}{x}\right)^{13-r} (-2x)^r$	5
			•[2] simplify powers of x **OR** coefficients **and** signs [2,5]	•[2] $(3)^{13-r}(-2)^r$ or x^{2r-13}	
			•[3] state simplified general term (completes simplification) [2,5]	•[3] ${}^{13}C_r (3)^{13-r}(-2)^r x^{2r-13}$	
			•[4] determine value of r [3,4]	•[4] $2r-13=9 \Rightarrow r=11$	
			•[5] evaluate term [1,3]	•[5] $-1437696x^9$	

Notes:

1. Accept -1437696.
2. For •[1] accept the initial appearance of $\sum_{r=0}^{13} {}^{13}C_r \left(\frac{3}{x}\right)^{13-r} (-2x)^r$ as bad form. •[2] and •[3] are available only to candidates who simplify a general term correctly.
3. •[4] and •[5] are the only marks available to candidates who have not proceeded from a general term eg. an expansion using Pascal's Triangle. The required term must be explicitly identified in order for •[5] to be awarded.
4. Starting with ${}^{13}C_r \left(\frac{3}{x}\right)^{r} (-2x)^{13-r}$ leading to $r=2$ can also gain full credit.
5. Accept $\frac{1}{x^{13-2r}}$ when awarding •[2] or •[3].

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
4.			•¹ Construct augmented matrix	•¹ $\left(\begin{array}{ccc:c} 1 & 2 & 3 & 3 \\ 2 & -1 & 4 & 5 \\ 1 & -3 & 2\lambda & 2 \end{array}\right)$	4
			•² Use row operations to establish first two zero elements [1]	•² $\left(\begin{array}{ccc:c} 1 & 2 & 3 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & -5 & 2\lambda-3 & -1 \end{array}\right)$	
			•³ Establish third zero element **OR** recognise linear relationship between two rows [1,2]	•³ $\left(\begin{array}{ccc:c} 1 & 2 & 3 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 2\lambda-1 & 0 \end{array}\right)$ **OR** $2\lambda-3=-2$	
			•⁴ State value of λ [2]	•⁴ $\lambda=\frac{1}{2}$	

Notes:

1. Elementary row operations must be carried out correctly for •² and •³ to be awarded.
2. •⁴ is only available where a candidate's final matrix exhibits redundancy.
3. Disregard any working/statement subsequent to $\lambda=\frac{1}{2}$.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
5.			•[1] show true for $n=1$ [1]	•[1] LHS: $1(3-1)=2$ RHS: $1^2(1+1)=2$ So true for $n=1$	**4**
			•[2] assume true for $n=k$ [2] **AND** consider $n=k+1$	•[2] $\sum_{r=1}^{k} r(3r-1)=k^2(k+1)$ **and** $\sum_{r=1}^{k+1} r(3r-1)=$ $\ldots=\sum_{r=1}^{k} r(3r-1) + (k+1)(3(k+1)-1)$	
			•[3] correct statement of sum to $(k+1)$ terms using inductive hypothesis	•[3] $=k^2(k+1)+(k+1)(3k+2)$ $=(k+1)\left[k^2+3k+2\right]$ $=(k+1)(k+1)(k+2)$	
			•[4] express explicitly in terms of $(k+1)$ **or** achieve stated aim/goal [3,4] **AND** communicate	•[4] $=(k+1)^2((k+1)+1)$, thus if true for $n=k$ then true for $n=k+1$ but since true for $n=1$, then by induction true for all $n\in\mathbb{N}$	

Notes:

1. "RHS = 2, LHS = 2" and/or "True for $n=1$" are insufficient for the award of •[1]. A candidate must demonstrate evidence of substitution into both expressions.

2. For •[2] acceptable phrases include: "If true for..."; "Suppose true for…"; "Assume true for…". However, ***not*** acceptable: "Consider $n=k$", "assume $n=k$" and "True for $n=k$".
Allow if appears at conclusion.

3. Full marks are available to candidates who state an aim/goal earlier in the proof and who subsequently achieve the stated aim/goal.

4. Minimum acceptable form for •[4]: "Then true for $n=k+1$, but since true for $n=1$, then true for all n" or equivalent.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
6.			**Method 1**		6
			•[1] for either function: first derivative and two evaluations **OR** all three derivatives **OR** all four evaluations	•[1] $f(x)=\sin 3x$ $f(0)=0$ $f'(x)=3\cos 3x$ $f'(0)=3$ $f''(x)=-9\sin 3x$ $f''(0)=0$ $f'''(x)=-27\cos 3x$ $f'''(0)=-27$ $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3\ldots$	
			•[2] complete derivatives and evaluations **AND** substitute	•[2] $f(x)=3x-\frac{27}{3!}x^3$ $=3x-\frac{9}{2}x^3$	
			•[3] for second function: first derivative and two evaluations **OR** all three derivatives **OR** all four evaluations	•[3] $f(x)=e^{4x}$ $f(0)=1$ $f'(x)=4e^{4x}$ $f'(0)=4$ $f''(x)=16e^{4x}$ $f''(0)=16$ $f'''(x)=64e^{4x}$ $f'''(0)=64$	
			•[4] complete derivatives and evaluations **AND** substitute	•[4] $f(x)=1+4x+\frac{16x^2}{2}+\frac{64x^3}{6}$ $=1+4x+8x^2+\frac{32}{3}x^3$	
			•[5] multiply expressions	•[5] $e^{4x}\sin 3x=\left(3x-\frac{9}{2}x^3\ldots\right)\left(1+4x+8x^2+\frac{32}{3}x^3\ldots\right)$ $=24x^3-\frac{9}{2}x^3+12x^2+3x\ \ldots$	
			•[6] multiply out and simplify Note 2	•[6] $=3x+12x^2+\frac{39}{2}x^3\ \ldots$	

Question	Generic Scheme	Illustrative Scheme	Max Mark

Notes:

1. If a candidate chooses to use the product rule to obtain the Maclaurin series for $e^{4x}\sin 3x$ without first obtaining series for e^{4x} and $\sin 3x$ separately then only $\bullet^5$ and $\bullet^6$ are potentially available. In this instance for the award of $\bullet^5$ apply the same principle as that used to award $\bullet^1$ and $\bullet^3$.

$f(x)=e^{4x}\sin 3x$	$f(0)=0$
$f'(x)=4e^{4x}\sin 3x+3e^{4x}\cos 3x$	$f'(0)=3$
$f''(x)=7e^{4x}\sin 3x+24e^{4x}\cos 3x$	$f''(0)=24$
$f'''(x)=-44e^{4x}\sin 3x+117e^{4x}\cos 3x$	$f'''(0)=117$

2. At $\bullet^6$ the appearance of terms in x^4 or above should be disregarded.

Commonly Observed Responses:

			Generic Scheme	Illustrative Scheme	Max Mark
			__Method 2__		
			$\bullet^1$ state the Maclaurin expansion for $\sin x$ 1	$\bullet^1$ $\sin x = x-\frac{x^3}{3!}...$	
			$\bullet^2$ substitute	$\bullet^2$ $\sin 3x = 3x-\frac{(3x)^3}{3!}...$ $\sin 3x = 3x-\frac{9x^3}{2}...$	
			$\bullet^3$ state the Maclaurin expansion for e^x 1	$\bullet^3$ $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}...$	
			$\bullet^4$ substitute	$\bullet^4$ $e^{4x} = 1+4x+\frac{(4x)^2}{2!}+\frac{(4x)^3}{3!}...$ $e^{4x}=1+4x+8x^2+\frac{32x^3}{3}...$	
			$\bullet^5$ multiply expressions	$\bullet^5$ $e^{4x}\sin 3x=\left(1+4x+8x^2...\right)\left(3x-\frac{9x^3}{2}...\right)$	
			$\bullet^6$ multiply out and simplify	$\bullet^6$ $e^{4x}\sin 3x = 3x+12x^2+\frac{39x^3}{2}+....$	

Notes:

1. For a candidate who writes down $\sin 3x = 3x-\frac{(3x)^3}{3!}...$ without first writing down the series for $\sin x$ then $\bullet^1$ may be awarded. A similar principle may be applied to the awarding of $\bullet^3$ if required.
2. At $\bullet^6$ the appearance of terms in x^4 or above should be disregarded.

Commonly Observed Responses:

<table>
<tr><th colspan="3">Question</th><th>Generic Scheme</th><th>Illustrative Scheme</th><th>Max Mark</th></tr>
<tr><td>7.</td><td>(a)</td><td></td><td>•1 calculate determinant 1</td><td>•1 -2</td><td>1</td></tr>
<tr><td colspan="6">Notes:
1. If a candidate chooses to find A^{-1} then •1 is only available where '$\det A$' is clearly identified.</td></tr>
<tr><td colspan="6">Commonly Observed Responses:
$A^{-1}=\frac{1}{\det A}(...)$
$A^{-1}=\frac{1}{-2}(...)$
Award •1

$A^{-1}=\frac{1}{-2}(...)$
Do not award •1</td></tr>
<tr><td></td><td>(b)</td><td></td><td><u>Method 1</u>

•2 find A^2

•3 use an appropriate method

•4 write in required form and explicitly state values of p and q Note 1</td><td>•2 $A^2=\begin{pmatrix}4 & 0\\ \lambda & 1\end{pmatrix}$

•3 $A^2=\begin{pmatrix}2 & 0\\ \lambda & -1\end{pmatrix}+\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$
$A^2=A+2I$

•4 $p=1$ and $q=2$</td><td>3</td></tr>
<tr><td></td><td></td><td></td><td><u>Method 2</u>

•2 find A^2

•3 use an appropriate method

•4 write in required form and explicitly state values of p and q Note 1</td><td>•2 $A^2=\begin{pmatrix}4 & 0\\ \lambda & 1\end{pmatrix}$

•3 $A^2=p\begin{pmatrix}2 & 0\\ \lambda & -1\end{pmatrix}+q\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

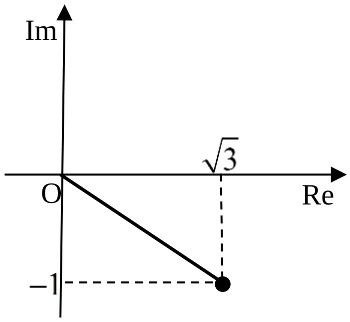
•4 $A^2=A+2I$
p = 1 and q = 2</td><td></td></tr>
<tr><td colspan="6">Notes:
1. $\begin{pmatrix}4 & 0\\ \lambda & 1\end{pmatrix}=\begin{pmatrix}2 & 0\\ \lambda & -1\end{pmatrix}+2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is acceptable for •4 provided the values of p and q are explicitly stated.</td></tr>
<tr><td colspan="6">Commonly Observed Responses:</td></tr>
</table>

Question			Generic Scheme	Illustrative Scheme	Max Mark
	(c)		•[5] square expression found in (b) [1,2,3] •[6] substitute for A^2 and complete process	•[5] $A^4 = (A+2I)^2$ $= A^2 + 4AI + 4I^2$ $= A + 2I + 4A + 4I$ •[6] $= 5A + 6I$	**2**

Notes:

1. •[5] may be obtained by squaring $\begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix}$ to give $\begin{pmatrix} 16 & 0 \\ 5\lambda & 1 \end{pmatrix}$ and identifying the coefficient of A as 5. This leads to •[6] using the same method as in (b).
2. Accept equivalent expressions eg $= A^2 + 4A + 4I$.
3. Candidates may calculate A^3 first so •[5] can be awarded for $A^3 = 3A + 2I$.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
8.	(a)		•¹ correctly plot z on Argand diagram 1,2,3,4	•¹ 	**1**

Notes:
1. Do not penalise the omission of the diagonal line.
2. Treat alternative axis labels as bad form (to include the case where there are no labels).
3. Accept a point labelled using coordinates: $\left(\sqrt{3},-1\right)$ and, in this instance, $\left(\sqrt{3},-i\right)$.
4. The minimum acceptable response for the award of •¹ is a point in quadrant 4 together with $\sqrt{3}$ and -1 (or $-i$).

Commonly Observed Responses:

	(b)		•² find modulus or argument 1,2,3,6	•² $\lvert w\rvert = 2a$ or $\arg(w) = -\frac{\pi}{6}$	**2**
			•³ complete and express in polar form 3,4,5,6	•³ $w = 2a\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$	

Notes:
1. For the award of •² and •³ accept any answer of the form $-\frac{\pi}{6}+2k\pi,\ k\in\mathbb{Z}$.
2. For the award of •² and •³ accept any answer of the form $(-30+360k)°,\ k\in\mathbb{Z}$.
3. A candidate who chooses to work in degrees can only be awarded •³ provided the degree symbol appears at some point within question 8.
4. Award •³ for $w = 2a\left(\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right)$.
5. At •³ do not accept $w = a\left[2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)\right]$.
6. Working subsequent to the appearance of $-\frac{\pi}{6}$ should be penalised where it leads to the use of an incorrect argument.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
	(c)		**Method 1**		**3**
			•⁴ process modulus	•⁴ $256a^8$	
			•⁵ process argument 1,2,3,4,5	•⁵ $\ldots\left(\cos\left(-\frac{8\pi}{6}\right)+i\sin\left(-\frac{8\pi}{6}\right)\right)$	
			•⁶ evaluate and express in form $ka^n\left(x+i\sqrt{y}\right)$	•⁶ $w^8=128a^8\left(-1+i\sqrt{3}\right)$	

Notes:

1. For the award of •⁵ accept any answer of the form $-\frac{4\pi}{3}+2k\pi,\ k\in\mathbb{Z}$.
2. For the award of •⁵ accept any answer of the form $\left(-240+360k\right)°,\ k\in\mathbb{Z}$.
3. A candidate who chooses to work in degrees can only be awarded •⁵ provided the degree symbol appears at some point within question 8.
4. Do not penalise unsimplified fractions.
5. Award •⁵ for $\ldots\left(\cos\frac{8\pi}{6}-i\sin\frac{8\pi}{6}\right)$.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
			Method 2		**3**
			•⁴ find w^2 correctly and attempt to find a higher power of w Note 1	•⁴ eg $w^2=a^2\left(2-2i\sqrt{3}\right)$ and $w^3=a^2\left(2-2i\sqrt{3}\right)\times a\left(\sqrt{3}-i\right)$.	
			•⁵ obtain w^4	•⁵ $w^4=a^4\left(-8-8i\sqrt{3}\right)$	
			•⁶ complete expansion and express in form $ka^n\left(x+i\sqrt{y}\right)$	•⁶ $w^8=128a^8\left(-1+i\sqrt{3}\right)$	

Notes:

1. Accept the omission of 'a' at •⁴ and •⁵ provided a^8 appears in the final answer.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
			Method 3		**3**
			•[4] write down full binomial expansion [1,2]	•[4] $\binom{8}{0}\left(\sqrt{3}\right)^8(-i)^0+\binom{8}{1}\left(\sqrt{3}\right)^7(-i)^1$ $+\binom{8}{2}\left(\sqrt{3}\right)^7(-i)^2\ldots+\binom{8}{8}\left(\sqrt{3}\right)^0(-i)^8$	
			•[5] simplifies individual terms	•[5] $81-216i\sqrt{3}-756+504i\sqrt{3}$ $+630-168i\sqrt{3}-84+8i\sqrt{3}+1$	
			•[6] complete expansion and express in form $ka^n\left(x+i\sqrt{y}\right)$	•[6] $w^8=128a^8\left(-1+i\sqrt{3}\right)$	

Notes:

1. For the award of •[4] a **full** expansion must be written out.
2. Accept the omission of 'a' at •[4] and •[5] provided a^8 appears in the final answer.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
9.			•¹ know to use integration by parts **and** start process [1,2,3]	•¹ $\frac{1}{8}x^8(\ln x)^2 - \ldots$	6
			•² correct choice of functions to differentiate and integrate **AND** application thereof [1,2,3]	•² $\ldots - \frac{1}{8}\int x^8 \times \frac{d}{dx}\left((\ln x)^2\right)dx$	
			•³ differentiate $(\ln x)^2$ [4]	•³ $\frac{1}{8}x^8(\ln x)^2 - \frac{1}{4}\int x^7(\ln x)\,dx$	
			•⁴ know to use second application and begin process [1,2,3,4]	•⁴ $\ldots - \left[\frac{1}{32}x^8(\ln x) - \frac{1}{32}\int x^8\left(\frac{1}{x}\right)dx\right]$	
			•⁵ complete second application	•⁵ $\ldots - \left[\frac{1}{32}x^8(\ln x) - \frac{1}{256}x^8\right]$	
			•⁶ simplify [5]	•⁶ $\frac{1}{8}x^8(\ln x)^2 - \frac{1}{32}x^8(\ln x) + \frac{1}{256}x^8 + c$	

Notes:

1. For candidates who attempt to integrate $(\ln x)^2$ and differentiate x^7 then •¹, •⁴ and •⁶ may be awarded but not •², •³ and •⁵.
2. Evidence of use of integration by parts would be the appearance of an attempt to integrate one term and differentiate the other.
3. For candidates who attempt to substitute for $\ln x$ eg $t = \ln x$ leading to $\int t^2 e^{8t}\,dt$ then
   - •¹ becomes available upon evidence of using integration by parts ie. $t^2.\frac{1}{8}e^{8t} - \ldots$
   - •⁶ is only available for a final answer expressed as a function of *x*.
4. For candidates who incorrectly differentiate $(\ln x)^2$ and do not require a second application of integration by parts, only •¹, •² and •⁶ are available.
5. Do not penalise the omission of "$+c$".

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
10.			•[1] give counterexample •[2] set up n Notes 1,2 •[3] consider expansion of n^3 Note 3 •[4] complete proof with conclusion [4]	•[1] eg. choose $p=7$ $2(7)+1=15$ and since $15=5\times 3$, hence not prime, statement is false •[2] $n=3a+1$, $a\in\mathbb{N}_0$ •[3] $n^3=27a^3+27a^2+9a+1$ •[4] $=3\left(9a^3+9a^2+3a\right)+1$ and statement such as "so n^3 has remainder 1 when divided by 3 $\therefore$ statement is true".	**4**

Notes:

1. Do not penalise the omission of $a\in\mathbb{N}_0$ in •[2].
2. Treat a statement such as $n=3n+1$ as bad form.
3. •[3] can only be awarded for the correct expansion of $(3a+1)^3$.
4. Minimum statement of conclusion in •[4] is "true".
5. Where a candidate invokes an incorrect use of proof by contradiction full credit may still be available provided all relevant steps are included.

Commonly Observed Responses:

<table>
<tr><th colspan="3">Question</th><th>Generic Scheme</th><th>Illustrative Scheme</th><th>Max Mark</th></tr>
<tr><td>11.</td><td></td><td></td><td><u>Method 1</u>

•¹ state differential equation [1,2]

•² state relationship or apply chain rule [3]

•³ find the rate of change of volume with respect to height [3]

•⁴ evaluate [4]</td><td>

•¹ $\frac{dh}{dt}=5$

•² $\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$

$V=h^3$

•³ $\frac{dV}{dh}=3h^2$

•⁴ $\frac{dV}{dt}=3h^2\times5=3(3)^2\times5=135\ \mathrm{cm^3\,s^{-1}}$</td><td>4</td></tr>
<tr><td></td><td></td><td></td><td><u>Method 2</u>

•¹ express volume as a function of time

•² find rate of change of volume with respect to time

•³ find value of t

•⁴ evaluate</td><td>

•¹ $V=125t^3$

•² $\frac{dV}{dt}=375t^2$

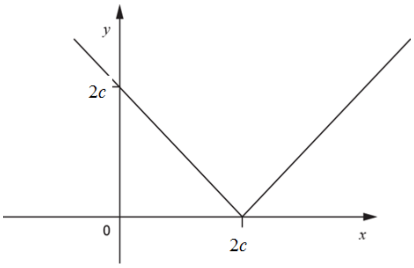
•³ $t=\frac{3}{5}$

•⁴ $\frac{dV}{dt}=375\left(\frac{3}{5}\right)^2=135\ \mathrm{cm^3\,s^{-1}}$</td><td></td></tr>
</table>

Notes:

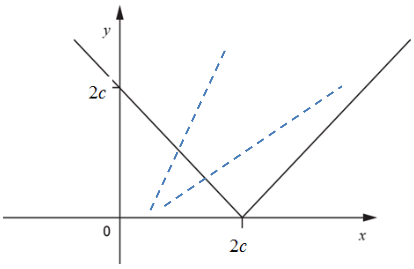
1. A candidate who assumes that only the height changes - and that the length and breadth are constant - can be awarded •¹ and •² only.
2. Where a candidate uses the wrong formula for the volume of a cube only •¹ and •² are available.
3. A candidate using Method 1 who writes $\frac{dV}{dt}=3h^2\frac{dh}{dt}$ can be awarded •² and •³.
4. To award •⁴ there must be evidence of substituting 3 and 5. Correct units must also be included.

Commonly Observed Responses:

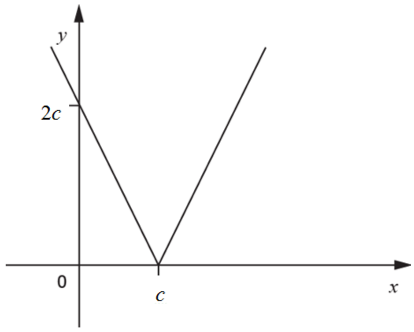
Question			Generic Scheme	Illustrative Scheme	Max Mark
12.	(a)		•¹ correct shape •² graph passes through $2c$ on the positive x- and y-axes	•¹,² 	**2**

Notes:

1. To award •², the second arm must be sketched to within 15° of the reflected angle.

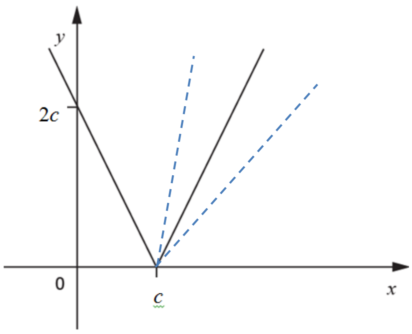


Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
	(b)		•³ graph of $y=\lvert 2f(x)\rvert$ passing through $2c$ on the positive y-axis 1 •⁴ correct shape (symmetrical V) meeting positive x-axis at c 2	•³,⁴ 	**2**

Notes:

1. For a candidate who sketches the graph of $y=2f(x)$ award •³ for showing a straight line passing through $(0,-2c)$.
2. To award •⁴, the second arm must be sketched to within 15° of the reflected angle.



Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
13.			•¹ correct application of partial fractions	•¹ $\frac{3x+32}{(x+4)(6-x)} = \frac{A}{x+4} + \frac{B}{6-x}$	**9**
			•² starts process	•² $3x+32 = A(6-x)+B(x+4)$	
			•³ calculate one value	•³ $A=2$	
			•⁴ calculate second value	•⁴ $B=5$	
			•⁵ re-state integral in partial fractions	•⁵ $\int_3^4 \left(\frac{2}{(x+4)} + \frac{5}{(6-x)}\right)dx$	
			•⁶ one term correctly integrated ¹	•⁶ $\left[2\ln\lvert x+4\rvert \ldots\right.$	
			•⁷ Integrate second term correctly ¹	•⁷ $\left.\ldots -5\ln\lvert 6-x\rvert\right]_3^4$	
			•⁸ substitute limits	•⁸ $\left(2\ln\lvert 4+4\rvert - 5\ln\lvert 6-4\rvert\right)$ $-\left(2\ln\lvert 3+4\rvert - 5\ln\lvert 6-3\rvert\right)$	
			•⁹ evaluate to expected form Note 3	•⁹ $=\ln\frac{486}{49}$	

Notes:

1. Do not penalise lack of modulus signs unless the candidate attempts to integrate $\frac{1}{x-6}$ rather than $\frac{1}{6-x}$.
2. Award maximum [8/9] for appropriate working leading to $\ln\frac{98}{243}$ (•⁹ lost)

   OR

   $\ln\frac{2048}{11907}$ (•⁷ lost).
3. Do not penalise unsimplified fractions in •⁹.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
14.	(a)		•[1] convert any two components of L_2 to parametric form [1]	•[1] two from $x=3-2\mu$, $y=8+\mu$, $z=-1+3\mu$	**5**
			•[2] two linear equations involving two distinct parameters	•[2] two from $4+3\lambda=3-2\mu$, $2+4\lambda=8+\mu$, $-7\lambda=-1+3\mu$	
			•[3] find parameter values	•[3] $\lambda=1,\ \mu=-2$	
			•[4] verify third component in **both** equations or equivalent	•[4] eg $z_1=-7\times1$ and $z_2=3(-2)-1$ therefore the lines intersect	
			•[5] find point of intersection	•[5] $(7,\ 6,\ -7)$	

Notes:

1. A candidate who uses λ as the second parameter can only be awarded •[1] unless this is rectified later in the question.
2. Do not penalise the omission of the statement 'therefore the lines intersect'.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
	(b)		•6 identify first direction vector [1,2,3]	•6 $\mathbf{d_1} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$	4
			•7 identify second direction vector [1,2,3]	•7 $\mathbf{d_2} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$	
			•8 calculate magnitudes and scalar product	•8 $\lvert\mathbf{d_1}\rvert = \sqrt{74}$, $\lvert\mathbf{d_2}\rvert = \sqrt{14}$ and $\mathbf{d_1} \cdot \mathbf{d_2} = -6 + 4 - 21 = -23$	
			•9 calculate obtuse angle [4,5]	•9 $\cos^{-1}\left(\dfrac{-23}{\sqrt{74}\sqrt{14}}\right) \approx 135\cdot 6°$	

Notes:

1. For $L_1 = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $L_2 = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ or equivalent, lose •6 but •7 is available (repeated error).
2. Do not penalise $\mathbf{L_1} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $\mathbf{L_2} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
3. For $L_1 : 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $L_2 : -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ or equivalent, •6 and •7 are both available.
4. For the award of •9 accept $136°$.
5. •9 is not available to candidates who calculate an obtuse angle correctly but who subsequently calculate an acute angle.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
15.			•[1] state auxiliary equation [1]	•[1] $m^2+5m+6=0$ $m=-3,\ m=-2$	**10**
			•[2] solve auxiliary equation and state complementary function [2,3]	•[2] $y=Ae^{-3x}+Be^{-2x}$	
			•[3] construct particular integral	•[3] $y=Cx^2+Dx+E$	
			•[4] differentiate particular integral	•[4] $\dfrac{dy}{dx}=2Cx+D$ and $\dfrac{d^2y}{dx^2}=2C$	
			•[5] calculate one coefficient of the particular integral	•[5] $C=2$	
			•[6] calculate remaining coefficients	•[6] $D=-3, E=1$ $y=Ae^{-3x}+Be^{-2x}+2x^2-3x+1$	
			•[7] differentiate general solution [3]	•[7] $\dfrac{dy}{dx}=-3Ae^{-3x}-2Be^{-2x}+4x-3$	
			•[8] construct equations using given conditions	•[8] $A+B=-7$ and $3A+2B=-6$ or equivalent	
			•[9] Find one coefficient	•[9] $A=8$ or $B=-15$	
			•[10] Find other coefficient **and** state particular solution	•[10] $y=8e^{-3x}-15e^{-2x}+2x^2-3x+1$	

Notes:

1. For •[1] do not penalise the omission of $'=0'$.
2. •[2] can be awarded if the Complementary Function appears later as part of the general solution, as opposed to being explicitly stated immediately after solving the Auxiliary Equation.
3. A candidate who obtains $m=2$ and $m=3$ from a correct auxiliary equation, leading to $y=20e^{3x}-27e^{2x}+2x^2-3x+1$ cannot gain •[2] but all other marks are available.
4. Where a candidate substitutes the given conditions into the Complementary Function to obtain values of A and B and then finds the particular integral correctly •[8] and •[9] are unavailable.

Commonly Observed Responses:

Question			Generic Scheme	Illustrative Scheme	Max Mark
16.			**Method 1** - working in minutes ($t=0$ at noon)		9
			•[1] construct integral equation Note 1	•[1] $\int \frac{1}{(T-T_F)}\,dT = \int -k\,dt$	
			•[2] integrate 2	•[2] $\ln(T-T_F) = -kt + c$	
			•[3] find constant, c	•[3] $\ln(9\cdot8-4) = -k(0)+c$ $c = \ln 5\cdot8$	
			•[4] substitute using given information 4	•[4] $\ln(6\cdot5-4) = -15k + \ln 5\cdot8$	
			•[5] find constant, k	•[5] $k = \frac{\ln 2\cdot5 - \ln 5\cdot8}{-15} = 0\cdot05610...$	
			•[6] substitute given condition	•[6] $\ln(25-4) = -0\cdot05610...t + \ln 5\cdot8$	
			•[7] know how to find time	•[7] $t = \frac{\ln 21 - \ln 5\cdot8}{-0\cdot05610...}$	
			•[8] calculate time	•[8] $t = -22\cdot93...$	
			•[9] state the time to the nearest minute 3	•[9] The liquid was placed in the fridge at 11:37 (am)	

Question			Generic Scheme	Illustrative Scheme	Max Mark
			Method 2 - working in minutes ($t=0$ when $T=25$)		
			•[1] construct integral equation Note 1	•[1] $\int \frac{1}{(T-T_F)}\,dT = \int -k\,dt$	
			•[2] integrate 2	•[2] $\ln(T-T_F) = -kt + c$	
			•[3] find constant, c.	•[3] $\ln(25-4) = -k(0)+c$, $c = \ln 21$	
			•[4] substitute using given information	•[4] $\ln(9\cdot8-4) = -k(t)+\ln 21$	
			•[5] know to use $t+15$ Note 5	•[5] appearance of $(t+15)$	
			•[6] use given condition	•[6] $\ln(6\cdot5-4) = -k(t+15)+\ln 21$	
			•[7] find constant, k Note 6	•[7] $k = -\frac{1}{15}\ln\left(\frac{2\cdot5}{5\cdot8}\right) = 0\cdot05610...$	
			•[8] calculate time	•[8] $t = \ln\left(\frac{21}{5\cdot8}\right) \div 0\cdot05610... = 22\cdot93$	
			•[9] state the time to the nearest minute 3	•[9] The liquid was placed in the fridge at 11:37 (am).	

Question			Generic Scheme	Illustrative Scheme	Max Mark
			Method 3 – working in hours ($t=0$ at midnight)		
			•[1] construct integral equation Note 1	•[1] $\int \frac{1}{(T-T_F)}\,dT = \int -k\,dt$	
			•[2] integrate 2	•[2] $\ln\left(T-T_F\right) = -kt + c$	
			•[3] use initial conditions	•[3] $\ln 5\cdot 8 = -12k + c$	
			•[4] interpret later time	•[4] $\ln 2\cdot 5 = -12\cdot 25k + c$	
			•[5] find constant, k	•[5] $\ln 5\cdot 8 - \ln 2\cdot 5 = 0\cdot 25k$ $k = 3\cdot 366...$	
			•[6] find the constant, c	•[6] $\ln\left(9\cdot 8-4\right) = -3\cdot 366... \times 12 + c$ $c = 42\cdot 15...$	
			•[7] know to find time	•[7] $\ln\left(25-4\right) = -3\cdot 366...t + 42\cdot 15$	
			•[8] calculate time	•[8] $t = \frac{42\cdot 15 - \ln 21}{3\cdot 366...}$ $= 11\cdot 62...$	
			•[9] state the time to the nearest minute 3	•[9] The liquid was placed in the fridge at 11:37 (am).	

Question			Generic Scheme	Illustrative Scheme	Max Mark
			Method 4 - working in minutes ($t=0$ when $T=25$)		
			•1 construct integral equation Note 1	•1 $\int \frac{1}{(T-T_F)}dT=\int -k\,dt$	
			•2 integrate 2	•2 $\ln(T-T_F)=-kt+c$ $T-T_F=e^{-kt+c}$ $T=Ae^{-kt}+T_F$ $T=Ae^{-kt}+4$	
			•3 use initial condition to calculate A	•3 $25=Ae^{-k(0)}+4 \therefore A=21$	
			•4 substitute using given information	•4 $9\cdot8=21e^{-kt}+4$	
			•5 know to use $t+15$ Note 7	•5 appearance of $(t+15)$	
			•6 substitute using given information	•6 $6\cdot5=21e^{-k(t+15)}+4$	
			•7 find constant, k	•7 $k=\dfrac{\ln\left(\frac{5\cdot8}{21}\right)-\ln\left(\frac{2\cdot5}{21}\right)}{15}=0\cdot0561...$	
			•8 calculate time	•8 $t=\ln\left(\frac{21}{5\cdot8}\right)\div0\cdot05610...=22\cdot93...$	
			•9 state the time to the nearest minute 3	•9 The liquid was placed in the fridge at 11:37 (am).	

Question	Generic Scheme	Illustrative Scheme	Max Mark

Notes:
General note:
Many candidates may use a combination of the given methods. For all methods the evidence for $\bullet^1$, $\bullet^2$, $\bullet^8$ and $\bullet^9$ is the same. To award $\bullet^3$ up to $\bullet^7$ note that:

two marks are awarded for using two different values of T
one mark is awarded for finding the constant of integration
one mark is awarded for finding or eliminating k (refer to Note **6**)
one mark is awarded for dealing with the elapsed time (noon until 12:15)

1. Do not penalise the omission of integral symbols at $\bullet^1$. (All Methods)
2. Do not penalise omission of "$+c$" at $\bullet^2$. However, it is necessary to access some later marks. (All Methods)
3. Where a candidate obtains an incorrect final answer because of earlier rounding, only $\bullet^9$ is unavailable. (All Methods)
4. For Method 1, if the candidate works in hours:
   - $\bullet^4$ $\ln(6\cdot5-4)=-0\cdot25k+\ln(5\cdot8)$
   - $\bullet^5$ $k=-4(\ln2\cdot5-\ln5\cdot8)=3\cdot366...$
   - $\bullet^6$ $\ln(25-4)=-3\cdot366...t+\ln5\cdot8$
   - $\bullet^7$ $t=\dfrac{\ln21-\ln5\cdot8}{-3\cdot366...}$
   - $\bullet^8$ $t=-0\cdot3822...$
5. For Method 2, if the candidate works in hours:
   - $\bullet^5$ appearance of $(t+0\cdot25)$
   - $\bullet^6$ $\ln(6\cdot5-4)=-k(t+0\cdot25)+\ln21$
   - $\bullet^7$ $k=-\dfrac{1}{0\cdot25}\ln\left(\dfrac{2\cdot5}{5\cdot8}\right)=3\cdot366...$
   - $\bullet^8$ $t=\ln\left(\dfrac{21}{5\cdot8}\right)\div0\cdot366...=0\cdot3822...$
6. In Method 2 $\bullet^7$ can be awarded for eliminating k.
7. For Method 4, if the candidate works in hours:
   - $\bullet^5$ appearance of $(t+0\cdot25)$
   - $\bullet^6$ $6\cdot5=21e^{-k(t+0\cdot25)}+4$
   - $\bullet^7$ $k=\dfrac{\ln\left(\dfrac{5\cdot8}{21}\right)-\ln\left(\dfrac{2\cdot5}{21}\right)}{0\cdot25}=3\cdot366...$
   - $\bullet^8$ $t=\ln\left(\dfrac{21}{5\cdot8}\right)\div3\cdot366...=0\cdot3822...$

Commonly Observed Responses:

[END OF MARKING INSTRUCTIONS]